

# Intersections of the Hermitian surface with irreducible quadrics in even characteristic

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## Abstract

We determine the possible intersection sizes of a Hermitian surface  $\mathcal{H}$  with an irreducible quadric of  $\text{PG}(3, q^2)$  sharing at least a tangent plane at a common non-singular point when  $q$  is even.

**Keywords:** Hermitian surface; quadrics; functional codes.

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## 1 Introduction

The study of intersections of geometric objects is a classical problem in geometry; see e.g. [11, 12]. In the case of combinatorial geometry, it has several possible applications either to characterize configurations or to construct new codes.

Let  $\mathcal{C}$  be a projective  $[n, k]$  linear code over  $\text{GF}(q)$ . It is always possible to consider the set of points  $\Omega$  in  $\text{PG}(k-1, q)$  whose coordinates correspond to the columns of any generating matrix for  $\mathcal{C}$ . Under this setup the problem of determining the minimum weight of  $\mathcal{C}$  can be reinterpreted, in a purely geometric setting, as finding the largest hyperplane sections of  $\Omega$ . More in detail, any codeword  $c \in \mathcal{C}$  corresponds to a linear functional evaluated on the points of  $\Omega$ ; see [18, 20]. For examples of applications of these techniques see [3, 4, 5].

Clearly, it is not necessary to restrict the study to hyperplanes. The higher weights of  $\mathcal{C}$  correspond to sections of  $\mathcal{C}$  with subspaces of codimension larger than 1; see [21] and also [16] for Hermitian varieties.

A different generalization consists in studying codes arising from the evaluation on  $\Omega$  of functionals of degree  $t > 1$ ; see [18]. These constructions yield, once more,

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linear codes, whose weight distributions depend on the intersection patterns of  $\Omega$  with all possible algebraic hypersurfaces of  $\text{PG}(k-1, q)$  of degree  $t$ .

The case of quadratic functional codes on Hermitian varieties has been extensively investigated in recent years; see [2, 7, 8, 9, 10, 17]. However, it is still an open problem to classify all possible intersection numbers and patterns between a quadric surface  $\mathcal{Q}$  in  $\text{PG}(3, q^2)$  and a Hermitian surface  $\mathcal{H} = \mathcal{H}(3, q^2)$ .

In [1], we determined the possible intersection numbers between  $\mathcal{Q}$  and  $\mathcal{H}$  in  $\text{PG}(3, q^2)$  under the assumption that  $q$  is an odd prime power and  $\mathcal{Q}$  and  $\mathcal{H}$  share at least one tangent plane. The same problem has been studied independently also in [6] for  $\mathcal{Q}$  an elliptic quadric; this latter work contains also some results for  $q$  even.

In this paper we fully extend the arguments of [1] to the case of  $q$  even. It turns out that the geometric properties being considered as well as the algebraic conditions to impose are different and more involved than those for the odd  $q$  case. Our main result is contained in the following theorem.

**Theorem 1.1.** *In  $\text{PG}(3, q^2)$ , with  $q$  even, let  $\mathcal{H}$  and  $\mathcal{Q}$  be respectively a Hermitian surface and an irreducible quadric with the same tangent plane at at least one common non-singular point  $P$ . Then, the possible sizes of the intersection  $\mathcal{H} \cap \mathcal{Q}$  are as follows.*

- For  $\mathcal{Q}$  elliptic:

$$q^3 - q^2 + 1, q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + q^2 + 1.$$

- For  $\mathcal{Q}$  a quadratic cone:

$$q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + 2q^2 - q + 1.$$

- For  $\mathcal{Q}$  hyperbolic:

$$q^2 + 1, q^3 - q^2 + 1, q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + q^2 + 1, \\ q^3 + 2q^2 - q + 1, q^3 + 2q^2 + 1, q^3 + 3q^2 - q + 1, 2q^3 + q^2 + 1.$$

We remark that, as we are dealing with irreducible quadrics in  $\text{PG}(3, q^2)$ , by quadratic cone (or, in short, cones) we shall always mean in dimension 3 the quadric projecting an irreducible conic contained in a plane  $\pi$  from a vertex  $V \notin \pi$ .

Our methods are algebraic in nature, based upon the  $\text{GF}(q)$ -linear representation of vector spaces over  $\text{GF}(q^2)$ , but in order to rule out some cases a more geometric and combinatorial approach is needed as well as some considerations on the action of the unitary groups.

For generalities on Hermitian varieties in projective spaces the reader is referred to [15, 19]. Basic notions on quadrics over finite fields are found in [14, 15].

## 2 Invariants of quadrics

In this section we recall some basic invariants of quadrics in even characteristic; the main reference for these results is [14, §1.1, 1.2], whose notation and approach we closely follow.

Recall that a quadric  $\mathcal{Q}$  in  $\text{PG}(n, q)$  is just the set of points  $(x_0, \dots, x_n) \in \text{PG}(n, q)$  such that  $F(x_0, \dots, x_n) = 0$  for some non-null quadratic form

$$F(x_0, \dots, x_n) = \sum_{i=0}^n a_i x_i^2 + \sum_{i < j} a_{ij} x_i x_j.$$

If there is no change of coordinates reducing  $F$  to a form in fewer variables, then  $\mathcal{Q}$  is *non-degenerate*; otherwise  $\mathcal{Q}$  is *degenerate*. The minimum number of indeterminates which may appear in an equation for  $\mathcal{Q}$  is the *rank* of the quadric, denoted by  $\text{rank } \mathcal{Q}$ ; see [13, §15.3].

Let consider the quadric  $\mathcal{Q}$  in  $\text{PG}(3, q)$  of equation  $\sum_{i=0}^3 a_i x_i^2 + \sum_{i < j} a_{ij} x_i x_j = 0$  and define

$$A := \begin{pmatrix} 2a_0 & a_{01} & a_{02} & a_{03} \\ a_{01} & 2a_1 & a_{12} & a_{13} \\ a_{02} & a_{12} & 2a_2 & a_{23} \\ a_{03} & a_{13} & a_{23} & 2a_3 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & a_{01} & a_{02} & a_{03} \\ -a_{01} & 0 & a_{12} & a_{13} \\ -a_{02} & -a_{12} & 0 & a_{23} \\ -a_{03} & -a_{13} & -a_{23} & 0 \end{pmatrix}$$

and

$$\alpha := \frac{\det A - \det B}{4 \det B}. \quad (1)$$

When  $q$  is even  $\det A$ ,  $\det B$  and  $\alpha$  should be interpreted as follows. In  $A$  and  $B$  we replace the terms  $a_i$  and  $a_{ij}$  by indeterminates  $Z_i$  and  $Z_{ij}$  and we evaluate  $\det A$ ,  $\det B$  and  $\alpha$  as rational functions over the integer ring  $\mathbb{Z}$ . Then we specialize  $Z_i$  and  $Z_{ij}$  to  $a_i$  and  $a_{ij}$ .

Actually, it turns out that  $\mathcal{Q}$  is non-degenerate if and only if  $\det A \neq 0$ ; see [14, Theorem 1.2]. By the same theorem, when  $q = 2^h$ , a non-degenerate quadric  $\mathcal{Q}$  of  $\text{PG}(3, q)$  is hyperbolic or elliptic according as

$$\text{Tr}_q(\alpha) = 0 \text{ or } \text{Tr}_q(\alpha) = 1,$$

respectively, where  $\text{Tr}_q$  denotes the absolute trace  $\text{GF}(q) \rightarrow \text{GF}(2)$  which maps  $x \in \text{GF}(q)$  to  $x + x^2 + x^{2^2} + \dots + x^{2^{h-1}}$ .

## 3 Some technical tools

In this section we are going to prove a series of lemmas that we will be useful in the proof of our main result, namely Theorem 1.1.

Henceforth, we shall assume  $q$  to be even;  $x, y, z$  will denote affine coordinates in  $\text{AG}(3, q^2)$  and the corresponding homogeneous coordinates will be  $J, X, Y, Z$ . The hyperplane at infinity of  $\text{AG}(3, q^2)$ , denoted as  $\Sigma_\infty$ , has equation  $J = 0$ .

Since all non-degenerate Hermitian surfaces of  $\text{PG}(3, q^2)$  are projectively equivalent, we can assume, without loss of generality,  $\mathcal{H}$  to have affine equation

$$z^q + z = x^{q+1} + y^{q+1}. \quad (2)$$

Since  $\text{PGU}(4, q)$  is transitive on  $\mathcal{H}$ , see [19, §35], we can also suppose that a point with common tangent plane to  $\mathcal{H}$  and  $\mathcal{Q}$  is  $P = P_\infty(0, 0, 0, 1) \in \mathcal{H}$ ; under these assumptions, the tangent plane at  $P$  to  $\mathcal{H}$  is  $\Sigma_\infty$ . Under the aforementioned assumptions,  $\mathcal{Q}$  has affine equation

$$z = ax^2 + by^2 + cxy + dx + ey + f \quad (3)$$

with  $a, b, c, d, e, f \in \text{GF}(q^2)$ . A direct computation proves that  $\mathcal{Q}$  is non-degenerate if and only if  $c \neq 0$ ; furthermore  $\mathcal{Q}$  is hyperbolic or elliptic according as the value of

$$\text{Tr}_{q^2}(ab/c^2)$$

is respectively 0 or 1. When  $c = 0$  and  $(a, b) \neq (0, 0)$ , the quadric  $\mathcal{Q}$  is a cone with vertex a single point.

Write now  $\mathcal{C}_\infty := \mathcal{Q} \cap \mathcal{H} \cap \Sigma_\infty$ . If  $\mathcal{Q}$  is elliptic, the point  $P_\infty$  is, clearly, the only point at infinity of  $\mathcal{Q} \cap \mathcal{H}$ , that is  $\mathcal{C}_\infty = \{P_\infty\}$ . The nature of  $\mathcal{C}_\infty$  when  $\mathcal{Q}$  is either hyperbolic or a cone, is detailed by the following lemma.

**Lemma 3.1.** *If  $\mathcal{Q}$  is a cone, then  $\mathcal{C}_\infty$  consists of either 1 point or  $q^2 + 1$  points on a line. When  $\mathcal{Q}$  is a hyperbolic quadric, then  $\mathcal{C}_\infty$  consists of either 1 point, or  $q^2 + 1$  points on a line or  $2q^2 + 1$  points on two lines. All cases may actually occur.*

*Proof.* As both  $\mathcal{H} \cap \Sigma_\infty$  and  $\mathcal{Q} \cap \Sigma_\infty$  split in lines through  $P_\infty$ , it is straightforward to see that the only possibilities for  $\mathcal{C}_\infty$  are those outlined above; in particular, when  $\mathcal{Q}$  is hyperbolic,  $\mathcal{C}_\infty$  consists of either 1 point or 1 or 2 lines. It is straightforward to see that all cases may actually occur, as given any two lines  $\ell, m$  in  $\text{PG}(3, q^2)$  there always exist at least one hyperbolic quadric containing both  $m$  and  $\ell$ . Likewise, given a line  $\ell \in \Sigma_\infty$  with  $P \in \ell$  there always is at least one cone with vertex  $V \in \ell$  and  $V \neq P$  meeting  $\Sigma_\infty$  just in  $\ell$ .  $\square$

We now determine the number of affine points that  $\mathcal{Q}$  and  $\mathcal{H}$  have in common.

**Lemma 3.2.** *The possible sizes of  $(\mathcal{H} \cap \mathcal{Q}) \setminus \Sigma_\infty$ , are either*

$$q^3 - q^2, q^3 - q^2 + q, q^3 - q, q^3, q^3 + q, q^3 + q^2 - q, q^3 + q^2$$

when  $\mathcal{Q}$  is elliptic, or

$$q^3 - q^2 + q, q^3 - q, q^3 + q, q^3 + q^2 - q,$$

when  $\mathcal{Q}$  is a cone, or

$$q^2, q^3 - q^2, q^3 - q^2 + q, q^3 - q, q^3, q^3 + q, q^3 + q^2 - q, q^3 + q^2, 2q^3 - q^2$$

when  $\mathcal{Q}$  is hyperbolic.

*Proof.* We have to study the following system of equations

$$\begin{cases} z^q + z = x^{q+1} + y^{q+1} \\ z = ax^2 + by^2 + cxy + dx + ey + f. \end{cases} \quad (4)$$

In order to solve (4), recover the value of  $z$  from the second equation and substitute it in the first. This gives

$$\begin{aligned} a^q x^{2q} + b^q y^{2q} + c^q x^q y^q + d^q x^q + e^q y^q + f^q + ax^2 + by^2 \\ + cxy + dx + ey + f = x^{q+1} + y^{q+1}. \end{aligned} \quad (5)$$

Consider  $\text{GF}(q^2)$  as a vector space over  $\text{GF}(q)$  and fix a basis  $\{1, \varepsilon\}$  with  $\varepsilon \in \text{GF}(q^2) \setminus \text{GF}(q)$ . Write any element in  $\text{GF}(q^2)$  as a linear combination with respect to this basis, that is, for any  $x \in \text{GF}(q^2)$  let  $x = x_0 + x_1 \varepsilon$ , where  $x_0, x_1 \in \text{GF}(q)$ . Thus, (5) can be studied as an equation over  $\text{GF}(q)$  in the indeterminates  $x_0, x_1, y_0, y_1$ . Analogously write also  $a = a_0 + \varepsilon a_1$ ,  $b = b_0 + \varepsilon b_1$  and so on.

As  $q$  is even, it is always possible to choose  $\varepsilon \in \text{GF}(q^2) \setminus \text{GF}(q)$  such that  $\varepsilon^2 + \varepsilon + \nu = 0$ , for some  $\nu \in \text{GF}(q) \setminus \{1\}$  and  $\text{Tr}_q(\nu) = 1$ . Then, also,  $\varepsilon^{2q} + \varepsilon^q + \nu = 0$ . Therefore,  $(\varepsilon^q + \varepsilon)^2 + (\varepsilon^q + \varepsilon) = 0$ , whence  $\varepsilon^q + \varepsilon + 1 = 0$ . With this choice of  $\varepsilon$ , (5) reads as

$$\begin{aligned} (a_1 + 1)x_0^2 + x_0 x_1 + [a_0 + (1 + \nu)a_1 + \nu]x_1^2 + (b_1 + 1)y_0^2 + y_0 y_1 \\ + [b_0 + (1 + \nu)b_1 + \nu]y_1^2 + c_1 x_0 y_0 + (c_0 + c_1)x_0 y_1 + (c_0 + c_1)x_1 y_0 \\ + [c_0 + (1 + \nu)c_1]x_1 y_1 + d_1 x_0 + (d_0 + d_1)x_1 + e_1 y_0 + (e_0 + e_1)y_1 + f_1 = 0. \end{aligned} \quad (6)$$

The solutions  $(x_0, x_1, y_0, y_1)$  of (6) correspond to the affine points of a (possibly degenerate) quadratic hypersurface  $\Xi$  of  $\text{PG}(4, q)$ . Recall that the number  $N$  of affine points of  $\Xi$  equals the number of points of  $\mathcal{H} \cap \mathcal{Q}$  which lie in  $\text{AG}(3, q^2)$ ; we shall use the results of [14] in order to actually count these points.

To this purpose, we first determine the number of points at infinity of  $\Xi$ . These points are those of the quadric  $\Xi_\infty$  of  $\text{PG}(3, q)$  with equation

$$\begin{aligned} f(x_0, x_1, y_0, y_1) = & (a_1 + 1)x_0^2 + x_0x_1 + [a_0 + (1 + \nu)a_1 + \nu]x_1^2 \\ & + (b_1 + 1)y_0^2 + y_0y_1 + [b_0 + (1 + \nu)b_1 + \nu]y_1^2 + c_1x_0y_0 \\ & + (c_0 + c_1)x_0y_1 + (c_0 + c_1)x_1y_0 + [c_0 + (1 + \nu)c_1]x_1y_1 \\ = & 0. \end{aligned} \quad (7)$$

Following the approach outlined in Section 2, write

$$A_\infty = \begin{pmatrix} 2(a_1 + 1) & 1 & c_1 & c_0 + c_1 \\ 1 & 2[a_0 + (1 + \nu)a_1 + \nu] & c_0 + c_1 & c_0 + (1 + \nu)c_1 \\ c_1 & c_0 + c_1 & 2(b_1 + 1) & 1 \\ c_0 + c_1 & c_0 + (1 + \nu)c_1 & 1 & 2[b_0 + (1 + \nu)b_1 + \nu] \end{pmatrix}.$$

As  $q$  is even, a direct computation gives  $\det A_\infty = 1 + c^{2(q+1)}$ . The quadric  $\Xi_\infty$  is non-degenerate if and only if  $\det A_\infty \neq 0$ , that is  $c^{q+1} \neq 1$ .

When  $\mathcal{Q}$  is a cone, namely  $c = 0$ , it turns out that  $\det A_\infty \neq 0$  and hence  $\text{rank } \Xi_\infty = 4$ .

Assume now  $\mathcal{Q}$  to be non-degenerate. If the equation of  $\Xi_\infty$  were to be of the form  $f(x_0, x_1, y_0, y_1) = (lx_0 + mx_1 + ny_0 + ry_1)^2$  with  $l, m, n, r$  over some extension of  $\text{GF}(q)$ , then  $c = 0$ ; this is a contradiction. So  $\text{rank } \Xi_\infty \geq 2$ .

We are next going to show that when  $\text{rank } \Xi_\infty = 2$ , that is  $\Xi_\infty$  splits into two planes, the quadric  $\mathcal{Q}$  is hyperbolic, that is  $\text{Tr}_{q^2}(ab/c^2) = 0$ . First observe that  $c^{q+1} = 1$  since the quadric  $\Xi_\infty$  is degenerate.

Consider now the following 4 intersections  $\mathcal{C}_0 : \Xi_\infty \cap [x_0 = 0]$ ,  $\mathcal{C}_1 : \Xi_\infty \cap [x_1 = 0]$ ,  $\mathcal{C}_2 : \Xi_\infty \cap [y_0 = 0]$ ,  $\mathcal{C}_3 : \Xi_\infty \cap [y_1 = 0]$ . Clearly, as  $\Xi_\infty$  is, by assumption, reducible in the union of two planes, all of these conics are degenerate; thus we get the following four formal equations

$$\begin{aligned} \frac{1}{2} \det \begin{pmatrix} 2[a_0 + (1 + \nu)a_1 + \nu] & c_0 + c_1 & c_0 + (1 + \nu)c_1 \\ c_0 + c_1 & 2(b_1 + 1) & 1 \\ c_0 + (1 + \nu)c_1 & 1 & 2[b_0 + (1 + \nu)b_1 + \nu] \end{pmatrix} &= 0, \\ \frac{1}{2} \det \begin{pmatrix} 2(a_1 + 1) & c_1 & c_0 + c_1 \\ c_1 & 2(b_1 + 1) & 1 \\ c_0 + c_1 & 1 & 2[b_0 + (1 + \nu)b_1 + \nu] \end{pmatrix} &= 0, \\ \frac{1}{2} \det \begin{pmatrix} 2(a_1 + 1) & 1 & c_0 + c_1 \\ 1 & 2[a_0 + (1 + \nu)a_1 + \nu] & c_0 + (1 + \nu)c_1 \\ c_0 + c_1 & [c_0 + (1 + \nu)c_1] & 2[b_0 + (1 + \nu)b_1 + \nu] \end{pmatrix} &= 0, \end{aligned}$$

$$\frac{1}{2} \det \begin{pmatrix} 2(a_1 + 1) & 1 & c_1 \\ 1 & 2[a_0 + (1 + \nu)a_1 + \nu] & c_0 + c_1 \\ c_1 & c_0 + c_1 & 2(b_1 + 1) \end{pmatrix} = 0.$$

Using the condition  $c^{q+1} = 1$ , these give

$$\begin{cases} a_0 + (1 + \nu)a_1 + (c_0^2 + c_1^2)b_0 + \nu(c_0^2 + c_1^2 + c_1^2\nu)b_1 = 0 \\ a_1 + c_1^2b_0 + (c_0^2 + \nu c_1^2)b_1 = 0 \\ (c_0^2 + c_1^2)a_0 + \nu(c_0^2 + c_1^2 + \nu c_1^2)a_1 + b_0 + (1 + \nu)b_1 = 0 \\ c_1^2a_0 + (c_0^2 + \nu c_1^2)a_1 + b_1 = 0. \end{cases} \quad (8)$$

If  $c_1 = 0$ , then  $c_0 = 1$  since  $c^{q+1} = 1$  and, solving (8), we obtain

$$\begin{cases} a_1 = b_1 \\ a_0 + a_1 + b_0 = 0. \end{cases} \quad (9)$$

Therefore  $\text{Tr}_{q^2}(ab/c^2) = \text{Tr}_{q^2}((a_0 + \varepsilon a_1)(a_0 + (\varepsilon + 1)a_1)) = \text{Tr}_{q^2}(a^{q+1}) = 0$  as  $a^{q+1} \in \text{GF}(q)$ .

Suppose now  $c_1 \neq 0$ . System (8) becomes

$$\begin{cases} a_0 = \left(\frac{c_0^2}{c_1^2} + \nu\right) a_1 + \frac{b_1}{c_1^2} \\ b_0 = \frac{a_1}{c_1^2} + \left(\frac{c_0^2}{c_1^2} + \nu\right) b_1; \end{cases} \quad (10)$$

hence,

$$\frac{ab}{c^2} = \frac{(a_1^2 + b_1^2)(c_1^2\nu + c_1^2\varepsilon + c_0^2) + a_1b_1(c_1^2\nu + c_1^2\varepsilon + c_0^2 + 1)^2}{c_1^4(c_0 + \varepsilon c_1)^2}.$$

Since  $\varepsilon^2 = \varepsilon + \nu$  and  $c_1^2\nu + c_0^2 + 1 = c_0c_1$ , we get

$$\frac{ab}{c^2} = \frac{a_1^2 + b_1^2}{c_1^4} + \frac{a_1b_1}{c_1^2} \in \text{GF}(q),$$

which gives  $\text{Tr}_{q^2}(ab/c^2) = 0$  once more. Hence if  $\text{rank } \Xi_\infty = 2$ , then  $\mathcal{Q}$  is hyperbolic.

When  $\text{rank } \Xi_\infty = 3$ , then  $\Xi_\infty$  is a cone comprising the join of a point to a conic.

Finally, when  $\Xi_\infty$  is non-degenerate, we can establish the nature of  $\Xi_\infty$  by computing the trace of  $\alpha$  as given by (1) where

$$B = \begin{pmatrix} 0 & 1 & c_1 & c_0 + c_1 \\ -1 & 0 & c_0 + c_1 & c_0 + (1 + \nu)c_1 \\ -c_1 & -(c_0 + c_1) & 0 & 1 \\ -(c_0 + c_1) & -c_0 - (1 + \nu)c_1 & -1 & 0 \end{pmatrix}.$$

Write  $\gamma = (1 + c^{q+1}) = (1 + c_0^2 + \nu c_1^2 + c_0 c_1)$ . A straightforward computation shows

$$\alpha = \frac{a_0 + a_1 + b_0 + b_1}{\gamma} + \frac{1}{\gamma^2}[(1 + \nu)(a_1^2 + b_1^2) + (c_0^2 + c_1^2 \nu)(a_0 b_1 + a_1 b_0) + a_0 a_1 + b_0 b_1 + a_0 b_0 c_1^2 + a_1 b_1 c_0^2]. \quad (11)$$

Thus, the following possibilities for  $N = |\Xi| - |\Xi_\infty| = |(\mathcal{H} \cap \mathcal{Q}) \cap \text{AG}(3, q^2)|$  may occur according as:

(C1)  $\text{rank } \Xi = 5$  and  $\text{rank } \Xi_\infty = 4$ ;

(C1.1)  $\Xi$  is a parabolic quadric and  $\Xi_\infty$  is a hyperbolic quadric ( $\det A_\infty \neq 0$  and  $\text{Tr}_q(\alpha) = 0$ ). Then,

$$N = (q + 1)(q^2 + 1) - (q + 1)^2 = q^3 - q.$$

(C1.2)  $\Xi$  is a parabolic quadric and the quadric  $\Xi_\infty$  is elliptic ( $\det A_\infty \neq 0$  and  $\text{Tr}_q(\alpha) = 1$ ). Then,

$$N = (q + 1)(q^2 + 1) - (q^2 + 1) = q^3 + q.$$

(C2)  $\text{rank } \Xi = 5$  and  $\text{rank } \Xi_\infty = 3$  ( $\det A_\infty = 0$ );

$\Xi$  is a parabolic quadric and the hyperplane at infinity is tangent to  $\Xi$ ; thus  $\Xi_\infty$  is a cone comprising the join of a point to a conic. Then,

$$N = (q + 1)(q^2 + 1) - (q^2 + q + 1) = q^3.$$

(C3)  $\text{rank } \Xi = 4$  and  $\text{rank } \Xi_\infty = 4$ ;

(C3.1)  $\Xi$  is a cone projecting a hyperbolic quadric of  $\text{PG}(3, q)$  and the quadric  $\Xi_\infty$  is hyperbolic ( $\det A_\infty \neq 0$  and  $\text{Tr}_q(\alpha) = 0$ ). Then,

$$N = q(q + 1)^2 + 1 - (q + 1)^2 = q^3 + q^2 - q.$$

(C3.2)  $\Xi$  is a cone projecting an elliptic quadric of  $\text{PG}(3, q)$  and the quadric  $\Xi_\infty$  is elliptic ( $\det A_\infty \neq 0$  and  $\text{Tr}_q(\alpha) = 1$ ). Then,

$$N = q(q^2 + 1) + 1 - (q^2 + 1) = q^3 - q^2 + q.$$

(C4)  $\text{rank } \Xi = 4$ ,  $\text{rank } \Xi_\infty = 3$ ;

(C4.1)  $\Xi$  is a cone projecting a hyperbolic quadric and  $\Xi_\infty$  is a cone comprising the join of a point to a conic. Then,

$$N = q(q + 1)^2 + 1 - [q(q + 1) + 1] = q^3 + q^2.$$



(C4.2)  $\Xi$  is a cone projecting an elliptic quadric and  $\Xi_\infty$  is a cone comprising the join of a point to a conic. Then,

$$N = q(q^2 + 1) + 1 - [q(q + 1) + 1] = q^3 - q^2.$$

(C5)  $\text{rank } \Xi = 4$  and  $\text{rank } \Xi_\infty = 2$ ;

(C5.1)  $\Xi$  is a cone projecting a hyperbolic quadric and  $\Xi_\infty$  is the union of two real planes. Then,

$$N = q(q + 1)^2 + 1 - (2q^2 + q + 1) = q^3.$$

(C5.2)  $\Xi$  is a cone projecting an elliptic quadric and  $\Xi_\infty$  the union of two planes defined over  $\text{GF}(q^2)$ . Then,

$$N = q(q^2 + 1) + 1 - (q + 1) = q^3.$$

(C6)  $\text{rank } \Xi = \text{rank } \Xi_\infty = 3$ ;

$\Xi$  is the join of a line to a conic and  $\Xi_\infty$  is a cone comprising the join of a point to a conic. Then,

$$N = q^3 + q^2 + q + 1 - (q^2 + q + 1) = q^3.$$

(C7)  $\text{rank } \Xi = 3$ ,  $\text{rank } \Xi_\infty = 2$ ;

(C7.1)  $\Xi$  is the join of a line to a conic whereas  $\Xi_\infty$  is a pair of planes over  $\text{GF}(q)$ . Then,

$$N = q^3 + q^2 + q + 1 - (2q^2 + q + 1) = q^3 - q^2.$$

(C7.2)  $\Xi$  is the join of a line to a conic whereas  $\Xi_\infty$  is a line. Then,

$$N = q^3 + q^2 + q + 1 - q - 1 = q^3 + q^2.$$

(C8)  $\text{rank } \Xi = \text{rank } \Xi_\infty = 2$ ;

(C8.1)  $\Xi$  is a pair of solids and  $\Xi_\infty$  is a pair of planes over  $\text{GF}(q)$ . Then,

$$N = 2q^3 + q^2 + q + 1 - (2q^2 + q + 1) = 2q^3 - q^2.$$

(C8.2)  $\Xi$  is a plane and  $\Xi_\infty$  is a line. Then,

$$N = q^2 + q + 1 - (q + 1) = q^2.$$

□

Now we are going to use the same group theoretical arguments as in [1, Lemma 2.3] in order to be able to fix the values of some of the parameters in (6) without losing in generality.

**Lemma 3.3.** *If  $\mathcal{Q}$  is a hyperbolic quadric, we can assume without loss of generality:*

1.  $b = 0$ , and  $a^{q+1} \neq c^{q+1}$  when  $\mathcal{C}_\infty$  is just the point  $P_\infty$ ;
2.  $b = 0$ ,  $a = c$  when  $\mathcal{C}_\infty$  is a line;
3.  $b = \beta a$ ,  $c = (\beta + 1)a$ ,  $a \neq 0$  and  $\beta^{q+1} = 1$ , with  $\beta \neq 1$  when  $\mathcal{C}_\infty$  is the union of two lines.

*If  $\mathcal{Q}$  is a cone, we can assume without loss of generality:*

1.  $b = 0$  when  $\mathcal{C}_\infty$  is a point;
2.  $a = b$  when  $\mathcal{C}_\infty$  is a line.

*Proof.* Let  $\Lambda$  be the set of all lines of  $\Sigma_\infty$  through  $P_\infty$ . The action of the stabilizer  $G$  of  $P_\infty$  in  $\text{PGU}(4, q)$  on  $\Lambda$  is the same as the action of  $\text{PGU}(2, q)$  on the points of  $\text{PG}(1, q^2)$ . This can be easily seen by considering the action on  $\text{PGU}(2, q)$  on the line  $\ell$  spanned by  $(0, 1, 0, 0)$  and  $(0, 0, 1, 0)$  fixing the equation  $X^{q+1} + Y^{q+1} = 0$ . Indeed, if  $M$  is a  $2 \times 2$  matrix representing any  $\sigma \in \text{PGU}(2, q)$ , then  $M' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & M_\sigma & 0 \\ 0 & 0 & 1 \end{pmatrix}$  represents an element of  $\text{PGU}(4, q)$  fixing  $P_\infty = (0, 0, 0, 1)$ . The action of  $\text{PGU}(2, q)$  on  $\ell$  is analyzed in detail in [19, §42]. So, we see that the group  $G$  has two orbits on  $\Lambda$ , say  $\Lambda_1$  and  $\Lambda_2$  where  $\Lambda_1$  consists of the totally isotropic lines of  $\mathcal{H}$  through  $P_\infty$  while  $\Lambda_2$  contains the remaining  $q^2 - q$  lines of  $\Sigma_\infty$  through  $P_\infty$ . Furthermore,  $G$  is doubly transitive on  $\Lambda_1$  and the stabilizer of any  $m \in \Lambda_1$  is transitive on  $\Lambda_2$ .

Let now  $\mathcal{Q}_\infty = \mathcal{Q} \cap \Sigma_\infty$ . If  $\mathcal{Q}$  is hyperbolic and  $\mathcal{C}_\infty = \{P_\infty\}$  we can assume  $\mathcal{Q}_\infty$  to be the union of the line  $\ell : J = X = 0$  and another line, say  $u : J = aX + cY = 0$  with  $a^{q+1} \neq c^{q+1}$ . Thus,  $b = 0$ .

Otherwise, up to the choice of a suitable element  $\sigma \in G$ , we can always take  $\mathcal{Q}_\infty$  as the union of any two lines in  $\{\ell, s, t\}$  where

$$\ell: J = X = 0, \quad s: J = X + Y = 0, \quad t: J = X + \beta Y = 0$$

with  $\beta^{q+1} = 1$  and  $\beta \neq 1$ .

Actually, when  $\mathcal{C}_\infty$  contains just one line we take  $\mathcal{Q}_\infty : X(X + Y) = 0$ , while if  $\mathcal{C}_\infty$  is the union of two lines we have  $\mathcal{Q}_\infty : (X + Y)(X + \beta Y) = 0$ . When  $\mathcal{Q}$  is a cone, we get either  $\mathcal{Q}_\infty : X^2 = 0$  or  $\mathcal{Q}_\infty : (X + Y)^2 = 0$ . The lemma follows. □

In the next three lemmas we denote by  $\Xi$  the quadric of  $\text{PG}(4, q)$  of equation (6), whereas by  $\Xi_\infty$  its section at infinity that is, the quadric of  $\text{PG}(3, q)$  of equation (7).

**Lemma 3.4.** *Suppose  $\mathcal{Q}$  to be a hyperbolic quadric with  $\mathcal{C}_\infty$  the union of two lines. If  $\text{rank } \Xi_\infty = 2$ , then  $\Xi_\infty = \Pi_1 \cup \Pi_2$  is a plane pair over  $\text{GF}(q)$ .*

*Proof.* By Lemma 3.3 we can assume that  $b = \beta a$ ,  $c = (\beta + 1)a$ ,  $a \neq 0$  and  $\beta^{q+1} = 1$  with  $\beta \neq 1$ .

Furthermore, since  $\text{rank } \Xi_\infty = 2$ , we can write

$$f(x_0, x_1, y_0, y_1) = (lx_0 + mx_1 + ny_0 + ry_1)(l'x_0 + m'x_1 + n'y_0 + r'y_1); \quad (12)$$

for some values of  $l, m, n, r$  and  $l', m', n', r'$ . Then, the following must be satisfied:

$$\left\{ \begin{array}{l} ll' = a_1 + 1 \\ lm' + l'm = 1 \\ l'n + ln' = c_1 \\ l'r + lr' = c_0 + c_1 \\ mm' = a_0 + (1 + \nu)a_1 + \nu \\ mn' + nm' = c_0 + c_1 \\ mr' + rm' = c_0 + (1 + \nu)c_1 \\ nr' + rn' = 1 \\ nn' = b_1 + 1 \\ rr' = b_0 + (1 + \nu)b_1 + \nu. \end{array} \right. \quad (13)$$

If  $c_1 = 0$ , then  $c_0 = 1$  as  $c^{q+1} = 1$ ; in particular, as  $c = b + a$ , we have  $a_0 + b_0 = c_0 = 1$  and, consequently, as (9) holds we get  $a_1 = 1 = b_1$ . System (13) becomes

$$\left\{ \begin{array}{l} ll' = 0 \\ lm' + l'm = 1 \\ l'n + ln' = 0 \\ l'r + lr' = 1 \\ mm' = a_0 + 1 \\ mn' + nm' = 1 \\ mr' + rm' = 1 \\ nr' + rn' = 1 \\ nn' = 0 \\ rr' = b_0 + 1. \end{array} \right.$$

So, either  $l = 0$  or  $l' = 0$ . Suppose the former; then  $l' \neq 0$  and  $m = l'^{-1}$ . We also have  $n = 0$  and consequently  $r = l'^{-1}$ . It follows that  $\Pi_1$  is the plane of equation  $x_1 + y_1 = 0$ . In particular, both  $\Pi_1$  and, consequently,  $\Pi_2$  are defined over  $\text{GF}(q)$ . If  $l' = 0$ , an analogous argument leads to  $\Pi_2 : x_1 + y_1 = 0$  and, once more,  $\Xi_\infty$  splits into two planes defined over  $\text{GF}(q)$ .

Now suppose  $c_1 \neq 0$ . From (13) we get in particular

$$\begin{cases} ll' = a_1 + 1 \\ ln' + l'n = c_1 \\ nn' = b_1 + 1 \\ mm' = a_0 + (1 + \nu)a_1 + \nu \\ mr' + rm' = c_0 + (1 + \nu)c_1 \\ rr' = b_0 + (1 + \nu)b_1 + \nu \\ nr' + rn' = 1 \\ lm' + l'm = 1. \end{cases} \quad (14)$$

We obtain  $ll' + nn' = a_1 + b_1 = c_1 = ln' + l'n$  and  $mm' + rr' = c_0 + (1 + \nu)c_1 = mr' + rm'$ . Hence,

$$(l' + n')(l + n) = 0, \quad (m + r)(m' + r') = 0.$$

There are the following cases to consider, namely

1.  $l = n$  and  $m = r$ ;
2.  $l' = n'$  and  $m' = r'$ ;
3.  $l = n$  and  $m' = r'$ ;
4.  $l' = n'$  and  $m = r$ .

Suppose first  $l = n$  and  $m = r$ ; then  $n(n' + l') = c_1 \neq 0$ ; consequently,  $n = l \neq 0$  and also  $n' \neq l'$ . If  $m = 0$ , then  $\Pi_1$  has equation  $x_0 + x_1 = 0$  and is defined over  $\text{GF}(q)$ ; then also  $\Pi_2$  is defined over  $\text{GF}(q)$  and we are done.

Suppose now  $m \neq 0$  (and hence  $r \neq 0$ ). We claim  $l/m \in \text{GF}(q)$ . This would give that  $\Pi_1$  is defined over  $\text{GF}(q)$ , whence the thesis. From (14) we have

$$\begin{cases} n' = \frac{b_1+1}{n} \\ r' = \frac{b_0+(1+\nu)b_1+\nu}{r} \\ nr' + rn' = 1. \end{cases}$$

Replacing the values of  $n'$  and  $r'$  in the last equation we obtain

$$n^2(b_0 + (1 + \nu)b_1 + \nu) + r^2(b_1 + 1) + nr = 0; \quad (15)$$

if we consider

$$\begin{cases} ll' = \frac{a_1+1}{n} \\ lm' + lm' = 1 \\ mm' = a_0 + (1 + \nu)a_1 + \nu \end{cases}$$

a similar argument on  $l, l', m, m'$  gives

$$l^2(a_0 + (1 + \nu)a_1 + \nu) + m^2(a_1 + 1) + lm = 0. \quad (16)$$

Since, by assumption,  $l = n$  and  $m = r$  we get

$$l^2(a_0 + b_0 + (1 + \nu)(a_1 + b_1)) + m^2(a_1 + b_1) = 0,$$

whence  $l^2/m^2 \in \text{GF}(q)$ . As  $q$  is even, this gives  $l/m \in \text{GF}(q)$ . The case  $l' = n'$  and  $m' = r'$  is analogous.

Now suppose  $l = n$  and  $m' = r'$ . As  $l' = \frac{(a_1+1)}{l}$  and  $n' + l' = \frac{c_1}{l}$ ,

$$n' = \frac{a_1 + c_1 + 1}{l} = \frac{b_1 + 1}{l}.$$

If  $m' = r' = 0$ , then  $\Pi_2$  has equation  $(a_1 + 1)x_0 + (b_1 + 1)y_0 = 0$  and, consequently, is defined over  $\text{GF}(q)$ . Suppose then  $m' = r' \neq 0$ . There are several subcases to consider:

- if  $m = 0$ , then  $m' = r' = l^{-1}$  and  $\Pi_2$  has equation  $(a_1 + 1)x_0 + x_1 + (b_1 + 1)y_0 + y_1 = 0$ , which is defined over  $\text{GF}(q)$ ;
- if  $r = 0$ , then  $m' = r' = n^{-1} = l^{-1}$  and we deduce, as above, that  $\Pi_2$  is defined over  $\text{GF}(q)$ ;
- finally, suppose  $m \neq 0 \neq r$ ; then  $b_0 + (1 + \nu)b_1 + \nu \neq 0$  and from (13) we get

$$m' = \frac{a_0 + (1 + \nu)a_1 + \nu}{m}, \quad r' = \frac{b_0 + (1 + \nu)b_1 + \nu}{r}.$$

Since  $m' = r'$  we deduce

$$\frac{m}{r} = \frac{a_0 + (1 + \nu)a_1 + \nu}{b_0 + (1 + \nu)b_1 + \nu} \in \text{GF}(q). \quad (17)$$

Observe that  $l' = (a_1 + 1)l^{-1}$  and also  $r' = (b_0 + (1 + \nu)b_1 + \nu)r^{-1}$ ; thus from (13) we obtain

$$l^2(b_0 + (1 + \nu)b_1 + \nu) + r^2(a_1 + 1) + (c_0 + c_1)lr = 0. \quad (18)$$

On the other hand, since  $lm' + l'm = 1$ ,

$$\frac{l}{m}(a_0 + (1 + \nu)a_1 + \nu) + \frac{m}{l}(a_1 + 1) = 1;$$

using (17) we obtain

$$\frac{l}{r}(b_0 + (1 + \nu)b_1 + \nu) + \frac{r}{l}(a_1 + 1) \left( \frac{a_0 + (1 + \nu)a_1 + 1}{b_0 + (1 + \nu)b_1 + 1} \right) = 1,$$

whence

$$l^2(b_0 + (1 + \nu)b_1 + \nu) + lr + r^2(a_1 + 1) \left( \frac{a_0 + (1 + \nu)a_1 + 1}{b_0 + (1 + \nu)b_1 + 1} \right) = 0; \quad (19)$$

thus, adding (18) to (19) we get

$$\frac{l}{r} = \frac{a_1 + 1}{c_0 + c_1 + 1} \left( \frac{a_0 + (1 + \nu)a_1 + 1}{b_0 + (1 + \nu)b_1 + 1} \right) \in \text{GF}(q)$$

and the plane  $\Pi_1$  is defined over  $\text{GF}(q)$ . The case  $l' = n'$  and  $m = r$  is analogous.

□

**Lemma 3.5.** *Suppose  $\mathcal{Q}$  to be a hyperbolic quadric a  $\mathcal{C}_\infty = \{P_\infty\}$ . If  $\text{rank } \Xi_\infty = 2$ , then  $\Xi_\infty$  is a line.*

*Proof.* By Lemma 3.3 we can assume  $b = 0$ . Since  $\text{rank } \Xi_\infty = 2$  we have  $c^{q+1} = 1$ . If  $c_1 = 0$  then by (9) we get  $a = 0$ . In the case in which  $c_1 \neq 0$  from (10) we again obtain  $a = 0$ . We have to show that  $\Xi_\infty$  is the union of two conjugate planes. In order to obtain this result, it suffices to prove that the coefficients  $l, m, n, r$  in (12) belong to some extension of  $\text{GF}(q)$  but are not in  $\text{GF}(q)$ . From (13) we have

$$\begin{cases} ll' = 1 \\ lm' + l'm = 1 \\ mm' = \nu; \end{cases}$$

thus,  $\frac{l\nu}{m} + \frac{m}{l} = 1$ ; hence  $\nu l^2 + lm + m^2 = 0$ . Since  $\text{Tr}_q(\nu) = 1$  this implies that  $\frac{l}{m} \notin \text{GF}(q)$ . □

Recall that a quadric  $\mathcal{Q}$  meeting a Hermitian surface  $\mathcal{H}$  in at least 3 lines of a regulus is permutable with  $\mathcal{H}$ ; see [13, §19.3, pag. 124]. In particular, in this case the tangent planes at each of the common points of  $\mathcal{Q}$  and  $\mathcal{H}$  coincide, that is to say  $\mathcal{Q}$  and  $\mathcal{H}$  intersect nowhere transversally. We have the following statement.

**Lemma 3.6.** *Suppose  $\mathcal{Q}$  to be hyperbolic and  $\mathcal{C}_\infty$  to be the union of two lines. If  $\Xi_\infty$  is degenerate then  $\Xi$  cannot be a cone projecting a hyperbolic quadric.*

*Proof.* Suppose  $\Xi$  to be a cone projecting a hyperbolic quadric; then, by Lemma 3.2, Case (C5.1),  $|\mathcal{H} \cap \mathcal{Q}| = q^3 + 3q^2 + 1$ . Let  $\mathcal{R}$  be a regulus of  $\mathcal{Q}$  and denote by  $r_1, r_2, r_3$  respectively the numbers of 1-tangents,  $(q+1)$ -secants and  $(q^2+1)$ -secants to  $\mathcal{H}$  in  $\mathcal{R}$ . A direct counting gives

$$\begin{cases} r_1 + r_2 + r_3 = q^2 + 1 \\ r_1 + (q+1)r_2 + r_3(q^2+1) = q^3 + 3q^2 + 1. \end{cases} \quad (20)$$

By straightforward algebraic manipulations we obtain

$$qr_1 + (q-1)r_2 = q(q^2 - q - 1).$$

In particular,  $r_2 = qt$  with  $t \leq q$ . If it were  $t = q$ , then  $r_1 = -1$  — a contradiction; so  $r_2 \leq q(q-1)$ . Solving (20) in  $r_1$  and  $r_3$ , we obtain

$$r_3 = \frac{q^2 + 2q - r_2}{q} \geq \frac{q^2 + 2q - q^2 + q}{q} = 3.$$

In particular, there are at least 3 lines of  $\mathcal{R}$  contained in  $\mathcal{H}$ . This means that  $\mathcal{Q}$  is permutable with  $\mathcal{H}$ , see [13, §19.3, pag. 124] or [19, §86, pag. 154] and  $|\mathcal{Q} \cap \mathcal{H}| = 2q^3 + q^2 + 1$ , a contradiction.  $\square$

## 4 Proof of Theorem 1.1

We use the same setup as in the previous section. Lemma 3.2 gives the possible values of  $N$  that is, the intersection sizes of the quadric  $\mathcal{Q}$  and the non-singular Hermitian variety  $\mathcal{H}$  in  $\text{AG}(3, q^2)$ . Recall that we have computed  $N$  as the difference  $|\Xi| - |\Xi_\infty|$  where  $\Xi$  is the quadric of  $\text{PG}(4, q)$  of equation (6), whereas  $\Xi_\infty$  is its section at infinity.

In order to complete the proof of Theorem 1.1 we need to determine the nature of  $\mathcal{C}_\infty = \{\mathcal{Q} \cap \mathcal{H}\} \setminus \text{AG}(3, q^2)$  in all possible cases.

### 4.1 The elliptic case

Let  $\mathcal{Q}$  be an elliptic quadric. In this case  $\mathcal{C}_\infty = \{P_\infty\}$ . The possible sizes for the affine part of  $\mathcal{H} \cap \mathcal{Q}$  correspond to cases (C1), (C2), (C3), (C4) and (C6) of Lemma 3.2, whence

$$|\mathcal{Q} \cap \mathcal{H}| = N + 1 \in \{ q^3 - q^2 + 1, q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + q^2 + 1 \}.$$

## 4.2 The degenerate case

Suppose  $\mathcal{Q}$  to be a cone. Since  $c = 0$  in (3), we get that  $\Xi_\infty$  is non degenerate as  $\det A_\infty = 1$ ; hence, the size of the affine part of  $\mathcal{H} \cap \mathcal{Q}$  falls in one of cases (C1) or (C3) in Lemma 3.2. Here  $\mathcal{C}_\infty$  is either a point or one line.

If  $\mathcal{C}_\infty$  consists of just 1 point, then we can assume  $b = 0$  in (3) by Lemma 3.3. Thus (11) becomes  $\alpha = a_0 + a_1 + (1 + \nu)a_1^2 + a_0a_1$  and hence cases (C1) and (C3) in Lemma 3.2 may happen; so,

$$|\mathcal{Q} \cap \mathcal{H}| = N + 1 \in \{q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + q + 1, q^3 + q^2 - q + 1\}.$$

If  $\mathcal{C}_\infty$  consists of  $q^2 + 1$  points on a line, then we can assume  $a = b$  in (3) by Lemma 3.3. In this case  $\alpha = 0$ ; thus, only subcases (C1.1) of (C1) and (C3.1) of (C3) in Lemma 3.2 may occur. In particular,

$$|\mathcal{Q} \cap \mathcal{H}| = N + q^2 + 1 \in \{q^3 + q^2 - q + 1, q^3 + 2q^2 - q + 1\}.$$

## 4.3 The hyperbolic case

If  $\mathcal{Q}$  is a hyperbolic quadric, then we have three possibilities for  $\mathcal{C}_\infty$ , that is  $\mathcal{C}_\infty$  is either a point, or a line or the union of two lines.

If  $\mathcal{C}_\infty = \{P_\infty\}$ , then all cases (C1)–(C8) of Lemma 3.2 might occur. We are going to show that some subcases of Lemma 3.2 can be excluded.

When  $\text{rank } \Xi_\infty = 2$ , from Lemma 3.5 we have that subcases (C7.2) and (C8.1) cannot occur. So,

$$|\mathcal{Q} \cap \mathcal{H}| = N + 1 \in \{q^2 + 1, q^3 - q^2 + 1, q^3 - q^2 + q + 1, q^3 - q + 1, q^3 + 1, q^3 + q + 1, q^3 + q^2 - q + 1, q^3 + q^2 + 1\}.$$

Suppose now  $\mathcal{C}_\infty$  to be exactly one line. By Lemma 3.3 we can assume  $b = 0$  and  $a = c$  in (3).

When  $\Xi_\infty$  is non degenerate, only cases (C1) and (C3) may occur. Observe that (11) becomes

$$\alpha = \frac{c_0 + c_1}{\gamma} + \frac{1}{\gamma^2}[(1 + \nu)(c_1^2) + c_0c_1].$$

Since  $\gamma = c_0^2 + \nu c_1^2 + c_0c_1 + 1$  we have  $c_1^2 + \nu c_1^2 + c_1c_0 = \gamma + c_0^2 + c_1^2 + 1$ . Therefore

$$\alpha = \frac{(c_0 + c_1)}{\gamma} + \frac{c_0^2 + c_1^2}{\gamma^2} + \frac{1}{\gamma} + \frac{1}{\gamma^2}$$

and  $\text{Tr}_q(\alpha) = 0$ . Hence, subcases (C1.2) and (C3.2) of Lemma 3.2 cannot happen; so,

$$|\mathcal{Q} \cap \mathcal{H}| = N + q^2 + 1 \in \{q^3 + q^2 - q + 1, q^3 + 2q^2 - q + 1\}.$$



Assume now that  $\Xi_\infty$  is degenerate. Cases (C2) and (C4)–(C8) of Lemma 3.2 occur. We are going to show that  $\text{rank } \Xi_\infty = 3$ . Suppose, on the contrary,  $\text{rank } \Xi_\infty = 2$ . As we have seen in the proof of Lemma 3.2 if it were  $c_1 = 0$ , from (9) we would have  $a_1 = a_0 = 0$ , that is  $a = 0$ , which is impossible. So,  $c_1 \neq 0$ ; since  $b_1 = b_0 = 0$ , we would now have from (10)  $a_1 = a_0 = 0$ —again a contradiction.

Thus, only cases (C2) and (C4) in Lemma 3.2 might happen; in particular,

$$|\mathcal{Q} \cap \mathcal{H}| = N + q^2 + 1 \in \{q^3 + 1, q^3 + q^2 + 1, q^3 + 2q^2 + 1\}.$$

Finally, when  $\mathcal{C}_\infty$  consists of two lines, by Lemma 3.3 we can assume  $b = \beta a$ ,  $c = (\beta + 1)a$  where  $a \neq 0$  and  $\beta^{q+1} = 1$  in (3).

Suppose now  $\Xi_\infty$  to be non degenerate. Cases (C1) and (C3) of Lemma 3.2 occur.

From  $c = a + b$  we get  $c^{q+1} = a^{q+1} + a^q b + b^q a + b^{q+1}$ . Since  $b^{q+1} = a^{q+1}$ , we have  $c^{q+1} = a^q b + b^q a$ , that is

$$a_0 b_1 + a_1 b_0 = c_0^2 + \nu c_1^2 + c_0 c_1,$$

On the other hand, from  $c_0 = a_0 + b_0$  and  $c_1 = a_1 + b_1$  we obtain  $c_0 c_1 = a_0 a_1 + a_0 b_1 + a_1 b_0 + b_0 b_1$  that is

$$a_0 a_1 + b_0 b_1 = c_0^2 + \nu c_1^2.$$

Now  $(a_0 b_1 + a_1 b_0)(a_0 a_1 + b_0 b_1) = a_0 b_0(a_1^2 + b_1^2) + a_1 b_1(a_0^2 + b_0^2) = a_0 b_0 c_1^2 + a_1 b_1 c_0^2 = (c_0^2 + \nu c_1^2 + c_0 c_1)(c_0^2 + \nu c_1^2)$  and thus (11) becomes

$$\alpha = \frac{c_0 + c_1}{1 + c_0^2 + \nu c_1^2 + c_0 c_1} + \frac{(c_0 + c_1)^2}{(1 + c_0^2 + \nu c_1^2 + c_0 c_1)^2};$$

so,  $\alpha$  has trace 0.

Hence just subcases (C1.1) and (C3.1) in Lemma 3.2 may occur and

$$|\mathcal{Q} \cap \mathcal{H}| = N + 2q^2 + 1 \in \{q^3 + 2q^2 - q + 1, q^3 + 3q^2 - q + 1\}.$$

If  $\Xi_\infty$  is degenerate, then cases (C2) and (C4)–(C8) in Lemma 3.2 may happen. By Lemma 3.6 only subcases (C4.2) in (C4) and (C5.2) in (C5) are possible.

In particular when  $\text{rank } \Xi_\infty = 2$ , it follows from Lemma 3.4 that only subcases (C7.1) in (C7) and (C8.1) in (C8) may occur. Thus we get

$$|\mathcal{Q} \cap \mathcal{H}| = N + 2q^2 + 1 \in \{q^3 + q^2 + 1, q^3 + 2q^2 + 1, 2q^3 + q^2 + 1\}$$

and the proof is completed.

It is straightforward to see, by means of a computer aided computation for small values of  $q$  that all the cardinalities enumerated above may occur.

## 5 Extremal configurations

As in the case of odd characteristic, it is possible to provide a geometric description of the intersection configuration when the size is either  $q^2 + 1$  or  $2q^3 + q^2 + 1$ . These values are respectively the minimum and the maximum yielded by Theorem 1.1. and they can happen only when  $\mathcal{Q}$  is an hyperbolic quadric. Throughout this section we assume that the hypotheses of Theorem 1.1 hold, namely that  $\mathcal{H}$  and  $\mathcal{Q}$  share a tangent plane at some point  $P$ .

**Theorem 5.1.** *Suppose  $|\mathcal{H} \cap \mathcal{Q}| = q^2 + 1$ . Then,  $\mathcal{Q}$  is a hyperbolic quadric and  $\Omega = \mathcal{H} \cap \mathcal{Q}$  is an ovoid of  $\mathcal{Q}$ .*

*Proof.* By Theorem 1.1,  $\mathcal{Q}$  is hyperbolic. Fix a regulus  $\mathcal{R}$  on  $\mathcal{Q}$ . The  $q^2 + 1$  generators of  $\mathcal{Q}$  in  $\mathcal{R}$  are pairwise disjoint and each has non-empty intersection with  $\mathcal{H}$ ; so there can be at most one point of  $\mathcal{H}$  on each of them. It follows that  $\mathcal{H} \cap \mathcal{Q}$  is an ovoid. In particular, by the above argument, any generator of  $\mathcal{Q}$  through a point of  $\Omega$  must be tangent to  $\mathcal{H}$ . Thus, at all points of  $\Omega$  the tangent planes to  $\mathcal{H}$  and to  $\mathcal{Q}$  are the same.  $\square$

**Theorem 5.2.** *Suppose  $|\mathcal{H} \cap \mathcal{Q}| = 2q^3 + q^2 + 1$ . Then,  $\mathcal{Q}$  is a hyperbolic quadric permutable with  $\mathcal{H}$ .*

Theorem 5.2 can be obtained as a consequence of the analysis contained in [7, §5.2.1], in light of [13, Lemma 19.3.1].

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